Homework 3

Due: Friday, Oct. 16, 2009

1. Let the density for X be given by

$$f(x) = ce^{-|x|} \quad -\infty < x < \infty$$

(a) Find the value of c that makes this a density.

(b) Show that

$$\frac{1}{2}\int_{-\infty}^{\infty} |x|e^{-|x|}dx$$

exists.

(c) Find $E[X]$.

(d) Show that

$$M_X(s) = \frac{-1}{s^2 - 1} \quad -1 < s < 1$$

(e) Use $M_X(s)$ to find $E[X]$ and $E[X^2]$.

(f) Find $Var[X]$.

2. Show that for $k > 0$ and $\beta > 0$,

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{1}{\beta^k \Gamma(k)} x^{k-1} e^{-x/\beta} dx = 1$$

thereby showing that the function $f(x)$ is a density for a continuous random variable.

3. Among diabetics, the fasting blood glucose level X may be assumed to be approximately normally distributed with mean 106 milligrams per 100 milliliters and standard deviation 8 milligrams per 100 milliliters.

(a) Sketch a graph of the density for X. Indicate on this graph the probability that a randomly selected diabetic will have a blood glucose level between 90 and 122 mg / 100 ml. Find this probability.

(b) Find $P[X \leq 120\, mg/\ 100\, ml]$.

(c) Find the point that has the property that 25% of all diabetics have a fasting glucose level of this value or lower.

(d) If a randomly selected diabetic is found to have fasting blood glucose level in excess of 130, do you think there is cause for concern? Explain, based on the probability of this occurring naturally.

4. A random variable X has the density function $f(x) = c/(x^2 + 1)$, where $-\infty < x < \infty$.

(a) Find the value of the constant c.

(b) Find the probability that X^2 lies between 1/3 and 1.

(c) Find the distribution function corresponding to the density function.

5. The *rth moment of a random variable X about the mean μ*, also called the *rth central moment*, is defined as

$$m_r = E[(X - \mu)^r]$$

where $r = 0, 1, 2, \cdots$. It follows that $m_0 = 1, m_1 = 0$ and $m_2 = \sigma^2$, i.e. the second central moment or second moment about the mean is the variance.

The *rth moment of X about the origin* is defined as

$$\bar{m}_r = E[X^r]$$

where $r = 0, 1, 2, \cdots$.

Prove the relationship between these moments is given by

$$m_r = \bar{m}_r - \binom{r}{1}\bar{m}_{r-1}\mu + \cdots + (-1)^j \binom{r}{j}\bar{m}_{r-j}\mu^j + \cdots + (-1)^r \bar{m}_0 \mu^r$$

6. Suppose that X has a Poisson distribution with mean λt; and that Y has a gamma distribution with shape parameter $= k$ and rate parameter $= \lambda$ where k is a positive integer. Show that $P[X \geq k] = P[Y \leq t]$ by showing that both the left side and the right side of this equation can be regarded as the probability of the same event in a Poisson process in which the expected number of occurrences per unit of time is λ.

7. Let N be a geometric random variable with $S_N = \{0, 1, \cdots\}$

   (a) Find $P[N > k]$.

   (b) Find the cdf of N.

   (c) Find $P[N \text{ is an even number}]$

   (d) Find $P[N = k | N \leq m]$.

8. A communication channel accepts an arbitrary voltage input v and outputs a voltage $Y = v + N$, where N is a Gaussian random variable with mean 0 and variance $\sigma^2 = 1$. Suppose that the channel is used to transmit binary information as follows:

to transmit 0,	input -1
to transmit 1,	input +1

   The receiver decides a 0 was sent if the voltage is negative and a 1 otherwise. Find the probability of the receiver making an error if a 0 was sent, if a 1 was sent.

9. Suppose that we observe the times S_m that elapses until the occurrence of the m-th event. The times $X_1, X_2, \cdots, X_m$ between events are exponential random variables, so we must have

$$S_m = X_1 + X_2 + \cdots + X_m$$

S_m is the m-Erland random variable.

(a) Find the cdf of of S_m by integration of the pdf. Hint: Use integration by parts.

(b) Show that the derivative of the cdf obtained in (a) gives the pdf of an m-Erland random variable.