Solution to Homework 2

Due: Friday, Oct. 2, 2009

1. 1 Grafting, the uniting of the stem of one plant with the stem or root of another, is widely used commercially to grow the stem of one variety that produces fine fruit on the root system of another variety with a hardy root system. Most Florida sweet oranges grow on trees grafted to the root of a sour orange variety. The density for X, the number of grafts that fail in a series of five trials, is given by Table 1.

   (a) find $P[X = 5]$

   (b) Find the table for F (cdf)

   (c) Use F to find the probability that at most three grafts fail; that at least two grafts fail.

   (d) Use F to verify that the probability of exactly three failures is .03.

   (e) Find $E[X], E[X^2], Var[X], \sigma_X$.

   (f) What physical unit is associated with σ_X?

Table 1: Question 1

x	0	1	2	3	4	5
$f(x)$	.7	.2	.05	.03	.01	?

Solution: (a)

$$\sum_{k=0}^{5} P[X = k] = 1$$

$$\Rightarrow P[X = 5] = 1 - \sum_{k=0}^{4}$$

$$= 1 - .7 - .2 - .05 - .03 - .01 = 0.01$$

(b) $F(x) = P[X \leq x]$, thus we get the table for F as in Table 2:

Table 2: Solution to 1(b)

x	0	1	2	3	4	5
$F(x)$	.7	.9	.95	.98	.99	1

(c) The probability that at most three grafts fail is $P[X \leq 3]$, thus from the table for F we get

$$P[X \leq x] = F(x) = F(3) = .98$$

The probability that at least two grafts fail is $P[X \geq 2]$, since

$$P[X \geq 2] = 1 - P[X < 2] = 1 - P[X \leq 1] = 1 - F(1)$$

From the table for F, we have

$$1 - F(1) = 1 - .9 = .1$$

(d) The probability of exactly three failures is $P[X = 3]$, since

$$P[X = 3] = F(3) - F(2) = .98 - .95 = .03$$

it agrees with the table for f.

(e)

$$\begin{aligned}
E[X] &= \sum_{k=0}^{5} kP[X = k] \\
&= 0(.7) + 1(.2) + 2(.05) + 3(.03) + 4(.01) + 5(.01) = .48 \\
E[X^2] &= \sum_{k=0}^{5} k^2 P[X = k] \\
&= 0^2(.7) + 1^2(.2) + 2^2(.05) + 3^2(.03) + 4^2(.01) + 5^2(.01) \\
&= 1.08 \\
Var[X] &= E[X^2] - (E[X])^2 = 1.08 - (.48)^2 = .8496 \\
\sigma_X &= \sqrt{Var[X]} = \sqrt{.8496} = .9217
\end{aligned}$$

(f) The unit that is associated with σ_X is the same as the unit for X.

2. The probability that a wildcat well will be productive is 1/13. Assume that a group is drilling wells in various parts of the country so that the status of one well has no bearing on that of any other. Let X denote the number of wells drilled to obtain the first strike.

   (a) Verify that X is geometric and identify the value of the parameter p.

   (b) What is the exact expression for the density for X?

   (c) What are the numeric values of $E[X], E[X^2], \sigma^2, \sigma$?

   (d) Find $P[X \geq 2]$.

   (e) Suppose that 10 wells have been drilled and no strikes have occurred. What is the probability that at least 2 more wells must be drilled in order to obtain the first strike? What property can be used to answer this question?

   Solution: (a) (1) The drilling of a well (trial) results in a strike (success) or not a strike (failure); (2) Trials are identical and independent with

$p = 1/13$ for each well; (3) $X =$ the number of trials (wells drilled) before the first success (strike).

(b) The density for X is

$$P[X = k] = \begin{cases} \left(1 - \frac{1}{13}\right)^{k-1} \left(\frac{1}{13}\right), & k = 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$$

(c)

$$\begin{aligned} E[X] &= \frac{1}{p} = \frac{1}{1/13} = 13 \\ Var[X] &= \frac{q}{p^2} = \frac{12/13}{1/13^2} = 12 * 13 = 156 \\ E[X^2] &= Var[X] - (E[X])^2 = 12 * 13 - (13)^2 \\ \sigma &= \sqrt{12 * 13} = 12.49 \end{aligned}$$

(d)

$$\begin{aligned} P[X \geq 2] &= \sum_{k=2}^{\infty} P[X = k] = \sum_{k=2}^{\infty} q^{k-1} p \\ &= 1 - P[X \leq 1] = 1 - F(1) \\ &= 1 - q^0 p = 1 - p = 12/13 \end{aligned}$$

(e) Based on the memoryless property of a geometric random variable, we know that,

$$P[X \geq 12 | X \geq 10] = P[X \geq 2]$$

thus, the required probability is the same as what is obtained in (d).

3. Albino rats used to study the hormonal regulation of a metabolic pathway are injected with a drug that inhibits body synthesis of protein. The probability that a rat will die from the drug before the experiment is over is .2. If 10 animals are treated with the drug, how many are expected to die before the experiment ends? What is the probability that at least eight will survive? Would you be surprised if at least five died during the course of the experiment? Explain, based on the probability of this occurring.

   Solution: The number X of rats who will die before the experiment ends is a binomial random variable with $n = 10$ and $p = .2$. The number of rats who are expected to die is $\mu_X = np = 10(.2) = 2$.

   The probability that at least eight will survive is the probability that at most two will die, i.e.

$$\begin{aligned} P[X \leq 2] &= P[X=0] + P[X=1] + P[X=2] \\ &= \binom{10}{0}(.2)^0(.8)^{10} + \binom{10}{1}(.2)^1(.8)^9 + \binom{10}{2}(.2)^2(.8)^8 \\ &= .68 \end{aligned}$$

The probability that at least five die is

$$\begin{aligned} P[X \geq 5] &= \sum_{k=5}^{10} \binom{10}{k}(.2)^5(.8)^5 \\ &= 1 - P[X < 5] = 1 - P[X \leq 4] = 1 - 0.9672 = 0.0327 \end{aligned}$$

This probability is so small, thus I will be surprised if at least five died.

4. In humans, geneticists have identified two sex chromosomes, R and Y. Every individual has an R chromosome, and the persence of a Y chromosome distinguishes the individual as male. Thus the two sexes are characterized as RR (female) and RY (male). Color blindness is caused by a recessive allele on the R chromosome, which we denote by r. The Y chromosome has no bearing on color blindness. Thus relative to color blindness, there are three genotypes for females and two for males:

Table 3: Question 4

Female	Male
RR (normal)	RY (normal)
Rr (carrier)	rY (color-blind)
rr (color-blind)	

A child inherits one sex chromosome randomly from each parent.

(a) A carrier of color blindness parents a child with a normal male. Construct a tree to present the possible genotypes for the child. Use the tree to find the probability that a given child will be a color-blind male.

(b) If the couple has five children, what is the expected number of color-blind males? What is the probability that three or more will be color-blind males?

Solution: (a) A carrier of color blindness mother has genotype Rr. A normal father has genotype RY. Their child will have four possible genotypes:

- RR: normal female
- RY: normal male

- rR: carrier female
- rY: color-blind male

The probability of each genotype is 1/4. Thus the probability that a given child will be a color-blind male is 1/4.

(b) The expected number of color-blind males among 5 children is $np = 5(.25) = 1.25$. That is 1 child.

The probability that three or more will be color-blind males is

$$P[X \geq 3] = 1 - P[X < 3] = 1 - F(2) = 1 - 0.8965 = 0.1035$$

5. In scanning electron microscopy photography, a specimen is placed in a vacuum chamber and scanned by an electron beam. Secondary electrons emitted from the specimen are collected by a detector and an image is displayed on a cathode-ray tube. This image is photographed. In the past a 4- $\times$ 5-inch camera has been used. It is thought that a 35-millimeter camera can obtain the same clarity. This type of camera is faster and more economical than the 4- $\times$ 5-inch variety.

   (a) Photographs of 15 specimens are made using each camera system. These unmarked photographs are judged for clarity by an impartial judge. The judge is asked to select the better of the two photographs from each pair. Let X denote the number selected taken by a 35-mm camera. If there is really no difference in clarity and the judge is randomly selecting photographs, what is the expected value of X?

   (b) Would you be surprised if the judge selected 12 or more photographs taken by the 35-mm camera? Explain, based on the probability involved.

   (c) If $X \geq 12$, do you think that there is reason to suspect that the judge is not selecting the photographs at random?

   Solution: (a) Let X denote the number of the 15 photographs taken by a 35-mm camera that were selected by the judge as better. Then X is a binomial random variable with parameters $n = 15$ and $p = .5$.

$$E[X] = np = 15(.5) = 7.5$$

   (b) The probability of this event occurring is

$$P[X \geq 12] = 1 - F(11) = 1 - .9824 = .0176$$

   Therefore, I will be surprised if the judge selected 12 or more photographs taken by the 35-mm camera.

   (c) yes. The probability that the judge selects the photograph at random is so small (.0176) that there is a reason to suspect that the judge is not.

6. Let X be a Poisson random variable with parameter $\lambda = 10$.

(a) Find $E[X]$

(b) Find $Var[X]$

(c) Find σ_X.

(d) Find $P[X \leq 4]$

(e) Find $P[X < 4]$

(f) Find $P[X = 4]$

(g) Find $P[X \geq 4]$

(h) Find $P[4 \leq X \leq 9]$.

Solution: The probability density function is

$$P[X = k] = \frac{\lambda^k}{k!}e^{-\lambda}$$

(a) $E[X] = \lambda = 10$

(b) $Var[X] = \lambda = 10$

(c) $\sigma_X = \sqrt{10}$

(d) $P[X \leq 4] = F(4) = .0292$

(e) $P[X < 4] = P[X \leq 3] = F(3) = .0103$

(f) $P[X = 4] = .0189$

(g) $P[X \geq 4] = 1 - P[X < 4] = 1 - P[X \leq 3] = 1 - F(3) = 1 - .0103 = .9897$

(h) $P[4 \leq X \leq 9] = F(9) - F(3) = .4579 - .0103 = .4476$

7. In *Escherichia coli*, a bacterium often found in the human digestive tract, 1 cell in every 10^9 will mutate from streptomycin sensitivity to streptomycin resistance. This mutation can cause the individual involved to become resistant to the antibiotic streptomycin. In observing 2 billion (2×10^9) such cells, what is the probability that none will mutate? What is the probability that at least one will mutate?

Solution: Let X denote the number of cells that will mutate in 2×10^9 cells. Then X is a Poisson random variable with $\lambda = 2$. Thus the probability that none will mutate in 2 billion cells is

$$P[X = 0] = \frac{2^0}{0!}e^{-2} = e^{-2}$$

The probability that at least one will mutate is

$$P[X \geq 1] = 1 - P[X = 0] = 1 - e^{-2}$$