Solution to Homework 1

Due: Friday, Sept. 18, 2009

1. When an individual is exposed to radiation, death may ensue. Factors affecting the outcome are the size of the dose, the length and intensity of the exposure, and the biological makeup of the individual. The term LD_{50} is used to denote the dose that is usually lethal for 50% of the individuals exposed to it. Assume that in a nuclear accident 30% of the workers are exposed to the LD_{50} and die; 40% of the workers die; and 68% are exposed to the LD_{50} or die. What is the probability that a randomly selected worker is exposed to the LD_{50}? Use a Venn diagram to find the probability that a randomly selected worker is exposed to the LD_{50} but does not die. Find the probability that a randomly selected worker is not exposed to the LD_{50} but dies.

   Solution: Let A represent the event that a worker is exposed to radiation and B represent the event that a worker dies. Then the following probabilities are given:

   $$P[B] = 0.4$$
   $$P[A \cap B] = 0.3$$
   $$P[A \cup B] = 0.68$$

   Then, the probability that a randomly selected worker is exposed to the LD_{50} is $P[A]$:

   $$P[A] = P[A \cup B] - P[B] - P[A \cap B] = 0.68 - 0.4 + 0.3 = 0.58$$

   The probability that a randomly selected worker is exposed to the LD_{50} but does not die is $P[A \setminus B]$ is:

   $$P[A \setminus B] = P[A] - P[A \cap B] = 0.58 - 0.3 = 0.28$$

   The probability that a randomly selected worker is not exposed to the LD_{50} but dies is $P[B \setminus A]$:

   $$P[B \setminus A] = P[B] - P[B \cap A] = 0.4 - 0.30 = 0.1$$

2. A test has been developed to detect a particular type of arthritis in individuals over 50 years old. From a national survey it is known that approximately 10% of the individuals in this age group suffer from this form of arthritis. The proposed test was given to individuals with confirmed arthritis disease, and a correct test result was obtained in 85% of the cases. When the test was administered to individuals of the same age group who were known to be free of the disease, 4% were reported to have the disease. What is the probability that an individual has this disease given that the test indicates its presence?

Solution: Let A be the event that a person has the disease, $+$ be the event that the test result is positive. Then we are given

$$P[A] = 0.1, \quad P[+|A] = 0.85, \quad P[+|A^c] = 0.04$$

Then, based on the Bayes' Theorem, we have

$$\begin{aligned} P[A|+] &= \frac{P[+|A]P[A]}{P[+|A]P[A] + P[+|A^c]P[A^c]} \\ &= \frac{0.85(0.1)}{0.85(0.1) + 0.04(0.9)} \end{aligned}$$

3. Prove the De Morgan's law and illustrate by using a Venn diagram:

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c$$

Solution: For any $a \in (A \cap B)^c$, we have

$$\begin{aligned} &a \in (A \cup B)^c \\ &\Rightarrow a \notin A \cup B \\ &\Rightarrow a \notin A \text{ and } a \notin B \\ &\Rightarrow a \in A^c \text{ and } a \in B^c \\ &\Rightarrow a \in A^c \cap B^c \end{aligned}$$

Similarly, for any $a \in A^c \cup B^c$, we have

$$\begin{aligned} &a \in A^c \cup B^c \\ &\Rightarrow a \in A^c \text{ or } a \in B^c \\ &\Rightarrow a \notin A \text{ or } a \notin B \\ &\Rightarrow a \notin A \cap B \\ &\Rightarrow a \in (A \cap B)^c \end{aligned}$$

4. The ability to observe and recall details is important in science. Unfortunately, the power of suggestion can distort memory. A study of recall is conducted as follows: Subjects are shown a film in which a car is moving along a country road. There is no barn in the film. The subjects are then asked a series of questions concerning the film. Half the subjects are asked, "How fast was the car moving when it passed the barn?" The other half is not asked the question. Later each subject is asked: " Is there a barn in the film?" Of those asked the first question concerning the barn, 17% answer "yes"; only 3% of the others answer "yes." What is the probability that a randomly selected participant in this study claims to have seen the nonexistent barn? Is claiming to see the barn independent of being asked the first question about the barn?

Solution: Let A represent the event that a randomly selected participant claims to have seen the nonexistent barn, let B represent the event that the participant was asked the question "How fast was the car moving when it passed the barn?" and B^c represent the event that the participant was NOT asked this question. Then

$$\begin{aligned} P[A|B] &= .17 \\ P[A|B^c] &= .03 \\ P[B] = P[B^c] &= .5 \end{aligned}$$

Then, based on the law of total probability, we have

$$\begin{aligned} P[A] &= P[A|B]P[B] + P[A|B^c]P[B^c] = (.17)(.5) + (.03)(.5) = 0.1 \\ P[A \cap B] &= P[A|B]P[B] = (.17)(.5) = .085 \end{aligned}$$

Since

$$P[A]P[B] = (.1)(.5) = 0.05 \neq P[A \cap B]$$

Claiming to see the barn is thus NOT independent of being asked the first question about the barn. That is to say that being asked the first question about the barn did influence the answers to the question that the participant was asked.

5. A box contains 6 red balls, 4 white balls, and 5 blue balls. Three balls are drawn successively from the box. Find the probability that they are drawn in the order red, white, and blue if each ball is (a) replaced, (b) not replaced.

   Solution: (a) with replacement

   P[1st is red, second is white, and third is blue]
   $= P$[1st is red] P[2nd is white|1st is red] P[third is blue|1st is red and 2nd is white]
   $= \frac{6}{15} \frac{4}{15} \frac{5}{15}$

   (b) without replacement

   P[1st is red, second is white, and third is blue]
   $= P$[1st is red] P[2nd is white|1st is red] P[third is blue|1st is red and 2nd is white]
   $= \frac{6}{15} \frac{4}{14} \frac{5}{13}$

6. The probability that a unit of blood was donated by a paid donor is .67. If the donor was paid, the probability of contracting serum hepatitis from the unit is .0144. If the donor was not paid, this probability is .0012. A patient receives a unit of blood. What is the probability of the patient's contracting serum hepatitis from this source?

Solution: Let A denote the event that the blood is from a paid donor. A^c will then be the event that the blood is from a non-paid donor. The following is given

$$\begin{aligned} P[A] &= .67 \\ P[\text{contracting hepatitis}|A] &= .0144 \\ P[\text{contracting hepatitis}|A^c] &= .0012 \end{aligned}$$

Then, based on the law of total probability, we obtain

$$\begin{aligned} P[\text{contracting hepatitis}] &= P[\text{contracting hepatitis}|A]P[A] + P[\text{contracting hepatitis}|A^c]P[A^c] \\ &= (.0144)(.67) + (.0012)(1 - 0.67) \end{aligned}$$

7. Theorem: If an event A must result in one of the mutually exclusive events $A_1, A_2, \cdots, A_n$, then

$$P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + \cdots + P(A_n)P(A|A_n)$$

Prove this theorem.

Proof:

$$A = (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n)$$

Because $A_1, A_2, \cdots, A_n$ are mutually exclusive, we have

$$\begin{aligned} P[A] &= P[A \cap A_1] + P[A \cap A_2] + \cdots + P[A \cap A_n] \\ &= P[A_1]P[A|A_1] + P[A_2]P[A|A_2] + \cdots + P[A_n]P[A|A_n] \end{aligned}$$

8. Studies in population genetics indicate that 39% of the available genes for determining the Rh blood factor are negative. Rh negative blood occurs if and only if the individual has two negative genes. One gene is inherited independently from each parent. What is the probability that a randomly selected individual will have Rh negative blood?

Solution: For a randomly selected individual to have Rh negative blood, he/she needs to inherit two negative genes from the parents. Since the inheritance from the mother and father is independent, we have

P[inherit one Rh gene from the father and inherit one Rh gene from the mother]

$= P$[inherit one Rh gene from the father] P[inherit one Rh gene from the mother]

$= (.39)(.39)$

9. In how many ways can 7 people be seated at a round table if (a) they can sit anywhere, (b) 2 particular people must not sit next to each other?

Solution: Let 1 of them be seated anywhere. Then the remaining 6 people can be seated in 6!=720 ways, which is the total number of ways of arranging the 7 people in a circle.

(b) Consider the 2 particular people as one person. Then there are 6 people altogether and they can be arranged in 5! ways. But the 2 people considered as 1 can be arranged among themselves in 2! ways. Thus the number of ways of arranging 7 people at a round table with 2 prticular people sitting together = 5!2! = 240.

Then, the total number of ways in which 7 people can be seated at a round table so that the 2 particular people do not sit together = 720-240=480 ways.

10. A power network involves three substations A, B and C. Overloads at any of these substations might result in a blackout of the entire network. Past history has shown that if substation A alone experiences an overload, then there is a 1% chance of a network blackout. For stations B and C alone these percentages are 2% and 3%, respectively. Overloads at two or more substations simultaneously result in a blackout 5% of the time. During a heat wave there is a 60% chance that substation A alone will experience an overload. For stations B and C these percentages are 20% and 15%, respectively. There is a 5% chance of an overload at two or more substations simultaneously. During a particular heat wave, a blackout due to an overload occurred. Find that probability that the overload occurred at substation A alone; substation B alone; substation C alone; two or more substations simultaneously.

Solution: We are given

$$\begin{aligned}
P[\text{blackout}|A \text{ overload}] &= .01\\
P[\text{blackout}|B \text{ overload}] &= .02\\
P[\text{blackout}|C \text{ overload}] &= .03\\
P[\text{blackout}| \geq 2 \text{ workstations overload}] &= .05\\
P[A \text{ overload}] &= .6\\
P[B \text{ overload}] &= .2\\
P[C \text{ overload}] &= .15\\
P[\geq 2 \text{ workstations overload}]
\end{aligned}$$

Based on the Bayes' theorem, we have

$$\begin{aligned}
P[A \text{ overload}|\text{blackout}] &= \frac{P[\text{blackout}|A \text{ overload}]P[A \text{ overload}]}{P[\text{blackout}]}\\
P[B \text{ overload}|\text{blackout}] &= \frac{P[\text{blackout}|B \text{ overload}]P[B \text{ overload}]}{P[\text{blackout}]}\\
P[C \text{ overload}|\text{blackout}] &= \frac{P[\text{blackout}|C \text{ overload}]P[C \text{ overload}]}{P[\text{blackout}]}\\
P[\geq 2 \text{ overload}|\text{blackout}] &= \frac{P[\text{blackout}| \geq 2 \text{ overload}]P[\geq 2 \text{ overload}]}{P[\text{blackout}]}
\end{aligned}$$

where

$$\begin{aligned} P[\text{blackout}] &= P[\text{blackout}|A \text{ overload}]P[A \text{ overload}] \\ &+ P[\text{blackout}|B \text{ overload}]P[B \text{ overload}] \\ &+ P[\text{blackout}|C \text{ overload}]P[C \text{ overload}] \end{aligned}$$

11. A person places n different letters into n differently addressed envelops at random. Find the probability that at least one of the letters will arrive at the proper destination.

Solution: Let $A_1, A_2, \cdots, A_n$ denote the events that the 1st, 2nd, $\cdots$,, nth letter is in the correct envelope. Then the event that at least one letter is in the correct envelope is $A_1 \cup A_2 \cup \cdots \cup A_n$ and we want to find $P[A_1 \cup A_2 \cup \cdots \cup A_n]$. We know that

$$\begin{aligned} P[A_1 \cup A_2 \cup \cdots \cup A_n] &= \sum P[A_k] - \sum P[A_j \cap A_k] \\ + \sum P[A_i \cap A_j \cap A_k] &- \cdots + (-1)^{n-1} P[A_1 \cap A_2 \cap \cdots \cap A_n] \end{aligned}$$

where $\sum P[A_k]$ is the sum of the probabilities of A_k from 1 to n, $\sum P[A_j \cap A_k]$ is the sum of the probabilities of $A_j \cap A_k$ with j and k from 1 to n and $k > j$, etc. We have for example the following:

$$P[A_j] = \frac{1}{n} \quad \text{for } j = 1, 2, \cdots, n$$

since of the n envelopes only 1 will have the proper address. Also

$$P[A_1 \cap A_2] = P[A_1]P[A_2|A_1] = \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)$$

since if the 1st letter is in the proper envelope then only 1 of the remaining $n-1$ envelopes will be proper. In a similar way, we find

$$\begin{aligned} P[A_1 \cap A_2 \cap A_3] &= P[A_1]P[A_2|A_1]P[A_3|A_1 \cap A_2] \\ &= \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right) \end{aligned}$$

etc. Finally

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\cdots\left(\frac{1}{1}\right) = \frac{1}{n!}$$

Now in the sum $\sum P[A_j \cap A_k]$ there are $\binom{n}{2}$ terms all having the value given by $\left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)$. Similarly in $\sum P[A_i \cap A_j \cap A_k]$ there are $\binom{n}{3}$ terms all having the value given by $\left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$. Thus the required probability is

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = \binom{n}{1}\left(\frac{1}{n}\right) - \binom{n}{2}\left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)$$
$$+ \binom{n}{3}\left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$
$$- \cdots + (-1)^{n-1}\binom{n}{n}\left(\frac{1}{n!}\right)$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}$$

From calculus we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

so that

$$e^{-1} = 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots\right)$$

or

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots = 1 - e^{-1}$$

It follows that if n is large the required probability is very nearly $1 - e^{-1} = 0.6321$. This means that there is a good chance of at least one letter arriving at the proper destination. The result is remarkable in that the probability remains practically constant for all $n > 10$. Thus the probability that at least one letter will arrive at its proper destination is practically the same whether n is 10 or $10,000$.

12. How many 4-digit numbers can be formed with the 10 digits $0,1,2,3,\cdots,9$ if (a) repetitions are allowed, (b) repetitions are not allowed, (c) the last digit must be zero and repetitions are not allowed?

Solution: (a) The first digit can be any one of 9 (since 0 is not allowed). The second, third and fourth digits can be any one of 10. Then $9 \cdot 10 \cdot 10 \cdot 10 = 9000$ numbers can be formed.

(b) The first digit can be any one of 9 (any one but 0)

The second digit can be any one of 9.

The third digit can be any one of 8.

The fourth digit can be any one of 7.

Then, $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ numbers can be formed.

(c) The first digit can be chosen in 9 ways, the second in 8 ways and the third in 7 ways. Then $9 \cdot 8 \cdot 7 = 504$ numbers can be formed.