Midterm Exam

Wednesday, Oct. 14, 2009

1. A simplified model of the human blood-type system has four blood types: A, B, AB, and O. There are two antigens, anti-A and anti-B, that react with a person's blood in different ways depending on the blood type. Anti-A reacts with blood types A and AB, but not with B and O. Anti-B reacts with blood types B and AB, but not with A and O. Suppose that a person's blood is sampled and tested with the two antigens. Let A be the event that the blood reacts with anti-A, and let B be the event that it reacts with anti-B. Classify the person's blood type using the events A, B, and their complements.

   Solution: Blood type A reacts only with anti-A, so type A blood corresponds to AB^c. Type B blood reacts only with anti-B, so type B blood corresponds to A^cB. Type AB blood reacts with both, so AB characterizes type AB blood. Finally, type O reacts with neither antigen, so type O blood corresponds to the event A^cB^c.

2. Consider, once again, problem 1. Suppose that, for a given person, the probability of type O blood is 0.5, the probability of type A blood is 0.34, and the probability of type B blood is 0.12.

   (a) Find the probability that each of the antigens will react with this person's blood.

   (b) Find the probability that both antigens will react with this person's blood.

   Solution: First note that the probability of type AB blood is $1 - (0.5 + 0.34 + 0.12) = 0.04$.

   (a) The probability of blood reacting with anti-A is the probability that the blood is either type A or type AB. Since these are disjoint events, the probability is the sum of the two probabilities, namely $0.34 + 0.04 = 0.38$. Similarly, the probability of reacting with anti-B is the probability of being either type B or type AB, 0.12+0.04=0.16.

   (b) The probability that both antigens react is the probability of type AB blood, namely 0.14.

3. The United States Senate contains two senators from each of the 50 states. What is the probability that a group of 50 senators selected at random will contain one senator from each state?

   Solution: There are $\binom{100}{50}$ combinations that might be chosen. If the group is to contain one senator from each state, then there are two possible choices from each of the 50 state. Hence, the number of possible combinations comtaining one senator from each state is 2^{50}. The required probability is thus $2^{50}/\binom{100}{50}$.

4. The probability that any child in a certain family will have blue eyes is 1/4, and this feature is inherited independently by different children in the family. If there are five children in the family and it is known that at least one of these children has blue eyes, what is the probability that at least three of the children have blue eyes?

   Solution: The probability p_j that exactly j children will have blue eyes is

$$p_j = \binom{5}{j} \left(\frac{1}{4}\right)^j \left(\frac{3}{4}\right)^{5-j}, \quad \text{for } j = 0, 1, 2, \cdots, 5$$

   The desired probability is

$$\frac{p_3 + p_4 + p_5}{p_1 + p_2 + p_3 + p_4 + p_5}$$

5. A new test has been devised for detecting a particular type of cancer. If the test is applied to a person who has this type of cancer, the probability that the person will have a positive reaction is 0.95 and the probability that the person will have a negative reaction is 0.05. If the test is applied to a person who does not have this type of cancer, the probability that the person will have a positive reaction is 0.05 and the probability that the person will have a negative reaction is 0.95. Suppose that in the general population, one person out of every 100,000 people has this type of cancer. If a person selected at random has a positive reaction to the test, what is the probability that he has this type of cancer?

   Solution: The desired probability $P[\text{cancer}|+]$ can be calculated as follows:

$$\begin{aligned} P[\text{cancer}|+] &= \frac{P[+|\text{cancer}]P[\text{cancer}]}{P[+|\text{cancer}]P[\text{cancer}] + P[+|\text{no cancer}]P[\text{no cancer}]} \\ &= \frac{(0.00001)(0.95)}{(0.00001)(0.95) + (0.99999)(0.05)} = 0.00019 \end{aligned}$$

6. Suppose that a random variable X has a discrete distribution with the following probability density function:

$$f(x) = \begin{cases} cx & x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

   Determine the value of the constant c.

   Solution: The sum of the values of $f(x)$ must be equal to 1. Since $\sum_{x=1}^{5} f(x) = 15c$, we must have $c = 1/15$.

7. Suppose that a random variable X has a uniform distribution on the interval [0,1]. Show that the expectation of $1/X$ does not exist.

Solution:

$$E[\frac{1}{X}] = \int_0^1 \frac{1}{X} dx = -\lim_{x \to 0} \log(x) = \infty$$

Since the integral is not finite, $E[\frac{1}{X}]$ does not exist.

8. Suppose that X is a random variable for which the moment generating function is as follows:

$$M(s) = \frac{1}{4}(3e^s + e^{-s}), \quad \text{for } -\infty < s < \infty$$

Find the mean and variance of X.

Solution:

$$M'(s) = \frac{1}{4}(3e^s - e^{-s})$$
$$M''(s) = \frac{1}{4}(3e^s + e^{-s})$$

Therefore,

$$\mu = M's|_{s=0} = \frac{1}{2}$$
$$\sigma^2 = M''s|_{s=0} - \mu^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

9. If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determine the probability that out of 2000 individuals, more than 2 individuals will suffer a bad reaction (Hint: use Poisson approximation to a binomial distribution).

Solution: Let X denote the number of individuals suffering a bad reaction. X is Bernoulli distributed, but since bad reactions are assumed to be rare events, we can suppose that X is Poisson distributed, i.e.

$$P[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

where $\lambda = np = 2000(0.001) = 2$.

$$P[X > 2] = 1 - P[X = 0] - P[X = 1] - P[X = 2] = 1 - 5e^{-2} = 0.323$$

10. Glucoma is an eye disease that is manifested by high intraocular pressure. The distribution of intraocular pressure in the general population is approximately normal with mean 16 mm Hg and standard deviation 3 mm Hg. If the normal range for intraocular pressure is considered to be between 12 mm Hg and 20 mm Hg, then what percentage of the general

population would fall within this range? (Just write down the expression for this probability. You do NOT need to obtain the numerical value.)

Solution: Let X denote the intraocular pressure. then we want to calculate $P[12 \leq X \leq 20]$ with X having the following normal density

$$f_X(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-16}{3}\right)^2}$$

then

$$P[12 \leq X \leq 20] = \int_{12}^{20} f_X(x) dx$$